Bayes Regression

$$Y = X\beta + \varepsilon \qquad \varepsilon \sim N(0, \sigma^2 I)$$

Assume we know σ.

$$p(\beta) \sim N(\bar{\beta}, A^{-1})$$

$$p(\beta) = (2\pi)^{-\frac{p}{2}} |A|^{\frac{1}{2}} \exp\left\{-\tfrac{1}{2}(\beta-\bar{\beta})^T A (\beta-\bar{\beta})\right\}$$

$$(\beta-\bar{\beta})^T A (\beta-\bar{\beta})$$

$$= \beta^T A \beta - 2\beta^T A \bar{\beta} + \bar{\beta}^T A \bar{\beta}$$

$$L(\beta) \propto \exp\left\{-\frac{1}{2\sigma^2}(y - X\beta)^T (y - X\beta)\right\}$$

Need:

*

$$\frac{(y - X\beta)^T (y - X\beta)}{\sigma^2} + (\beta - \bar{\beta})^T A (\beta - \bar{\beta})$$

In the form

$$(\beta - \bar{\bar{\beta}})^T V^{-1} (\beta - \bar{\bar{\beta}})$$

$$\frac{(y - X\beta)^T (y - X\beta)}{\sigma^2}$$

$$= \frac{1}{\sigma^2} (y^T y - 2\beta X^T y + \beta^T X^T X \beta)$$

(*) =

$$\beta^T \frac{X^T X}{\sigma^2} \beta - 2\beta \frac{X^T y}{\sigma^2}$$

$$+ \beta^T A \beta - 2\beta A \bar{\beta}$$

$$+ c$$

$(*) =$

$$\beta^T \left[\frac{X^T X}{\sigma^2} + A \right] \beta$$

$$- 2\beta \left[\frac{X^T y}{\sigma^2} + A\bar{\beta} \right] + c$$

$$V \equiv \left[\frac{X^T X}{\sigma^2} + A \right]^{-1}$$

$(*) =$

$$\beta^T V^{-1} \beta - 2\beta V^{-1} V \left[\frac{X^T y}{\sigma^2} + A\bar{\beta} \right]$$

$$+ c$$

$$\beta \mid \sigma, y \sim N\left(\left[\frac{X^T X}{\sigma^2} + A \right]^{-1} \left(\frac{X^T y}{\sigma^2} + A\bar{\beta} \right), \left(\frac{X^T X}{\sigma^2} + A \right)^{-1} \right)$$

Note

"flat prior" $\Rightarrow A \approx 0$

$\sim \beta | \sigma, y \sim$

$N\left((X^T X)^{-1} X^T y,\ \sigma^2 (X^T X)^{-1} \right)$

plug-in $\hat{\sigma}$

$\beta_i \sim N(b_i, S_{b_i}^2)$

b_i: least square
S_{b_i}: usual $se(b_i)$.

Nota

$$Y_i \sim N(\mu, \sigma^2) \quad \text{i.i.d.}$$

$$Y_i = \mu + \varepsilon_i \quad \varepsilon_i \sim N(0, \sigma^2)$$

$$Y = \mathbb{1}\mu + \varepsilon$$

$$\mathbb{1} = (1, 1, \ldots, 1)^T, \quad \varepsilon \sim N(0, \sigma^2 I)$$

$$X^T X = n$$
$$X^T y = n\bar{y}$$

$$\mu \sim N(\bar{\mu}, \tau^2) \iff \bar{\beta} = \bar{\mu}, \quad A = \frac{1}{\tau^2}$$

$$\left(\frac{X^T X}{\sigma^2} + A\right)^{-1}\left(\frac{X^T y}{\sigma^2} + A\bar{\beta}\right)$$

$$= \frac{\frac{n}{\sigma^2}\bar{y} + \frac{1}{\tau^2}\bar{\mu}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}$$

Gibbs Sampler for Bayesian Regression

$$y = X\beta + \varepsilon \qquad \varepsilon \sim N(0, \sigma^2 I)$$

$$\beta \sim N(\bar{\beta}, A^{-1})$$

independent

$$\sigma^2 \sim \frac{\nu\lambda}{\chi^2_\nu}$$

$\vdots$

Gibbs

$$\beta \mid \sigma, y$$

$$\sigma \mid \beta, y$$

$\sigma \mid \beta, y$: $\quad \varepsilon = y - X\beta \sim N(0, \sigma^2)$ iid.