# Introduction to the Metropolis Algorithm

Again to get the basic idea we consider the discrete case with

$$p_{ij} > 0$$

## Time Reversible Markov Chains

Reverse time order, what is ?

$$P\{X_m = j \mid X_{m+1} = i_1, X_{m+2} = i_2, \ldots X_{m+k} = i_k\}$$

??

$$= \frac{P\{X_m = j, X_{m+1} = i_1, \ldots, X_{m+k} = i_k\}}{P\{X_{m+1} = i_1, \ldots, X_{m+k} = i_k\}}$$

$$= \frac{P\{X_m = j\} P\{X_{m+1} = i_1 \mid X_m = j\} P\{X_{m+2} = i_2, \ldots, X_{m+k} = i_k \mid X_m = j, X_{m+1} = i_1\}}{P\{X_{m+1} = i_1\} P\{X_{m+2} = i_2, \ldots, X_{m+k} = i_k \mid X_{m+1} = i_1\}}$$

$$= \frac{P\{X_m = j\} P_{ji_1}}{P\{X_{m+1} = i_1\}}$$

$$\equiv \frac{\pi_j P_{ji_1}}{\pi_{i_1}}$$

or $$\frac{\pi_j P_{ji}}{\pi_i}$$

So the reverse process is a Markov chain with

$$P^*_{ij} = \frac{\pi_j P_{ji}}{\pi_i}$$

# Time Reversible

A MC is TR if

$$P^*_{ij} = P_{ij}$$

**or** $$\pi_i P_{ij} = \pi_j P_{ji}$$

"the chance of seeing an $i \to j$ transition is the same as the chance of seeing a $j \to i$ transition"

Now suppose we have

$$x_i > 0$$

s.t. $\quad \sum x_i = 1$

$$x_i P_{ij} = x_j P_{ji}$$

then

$$\sum_i x_i P_{ij} = x_j \sum_i P_{ji} = x_j$$

$$xP = x$$

$$\Rightarrow \quad x = \pi$$

Example: $\quad P = \begin{bmatrix} \frac{1}{4} & 3/4 \\ 3/8 & 5/8 \end{bmatrix}$

$$\pi = [\tfrac{1}{3}, 2/3]$$

$$\pi_1 P_{12} = \tfrac{1}{3}(3/4) = 3/12$$

$$\pi_2 P_{21} = \tfrac{2}{3}(3/8) = \tfrac{6}{24} = 3/12$$

Example: ~~If $P_{ij} = P_{ji}$ $\pi \propto 1$~~

## Example

$$P = \begin{bmatrix} .05 & .9 & .05 \\ .05 & .05 & .9 \\ .9 & .05 & .05 \end{bmatrix}$$

$$\pi = \{ \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3} \}$$

$$\pi P = \pi$$

$$\pi_1 P_{12} = \tfrac{1}{3}\left(\tfrac{9}{10}\right) = 3/10$$

$$\pi_2 P_{21} = \tfrac{1}{3}(.05) \neq 3/10.$$

## Example

suppose $P_{ij} = P_{ji}$

$$\Rightarrow \pi = \alpha \underline{1}.$$

## Metropolis Hastings Alg

Goal: construct a MC
whose stationary
distribution is $\pi$

know: $\pi_i / \pi_j \quad \forall\, i, j$

Let $q_{ij}$ be a transition matrix

Define a new MC based on
$q_{ij}$ and $\pi_i / \pi_j$ as follows.

Suppose you are at $i$.
Draw $j$ from $q$.
Compute $\alpha(i,j) = \min\left\{1, \dfrac{\pi_j q_{ji}}{\pi_i q_{ij}}\right\}$

With prob $\alpha$ go to $j$.
With prob $1-\alpha$ stay at $i$
(repeat $i$)

Note:
If $j=i$ you repeat

then

$$p_{ij} = q_{ij}\,\alpha(i,j) \qquad i \neq j$$

$$\pi_i p_{ij} = \pi_i q_{ij} \min\left\{1, \frac{\pi_j q_{ji}}{\pi_i q_{ij}}\right\}$$

$$= \min\{\pi_i q_{ij}, \pi_j q_{ji}\}$$

$$= \pi_j p_{ji}$$

P is reversible with stationary distribution $= \pi$

Example

$$Q = \begin{bmatrix} \frac{1}{4} & 3/4 \\ 3/4 & 1/4 \end{bmatrix} \qquad \pi_Q = \left[\tfrac{1}{2}, \tfrac{1}{2}\right]$$

Let $\pi = \left[\tfrac{1}{3}, 2/3\right]$

$$P_{12} = 3/4 \min\{1, \pi_2/\pi_1\}$$

$$= 3/4 \min\{1, 2\} = 3/4$$

$$\Rightarrow \quad P_{11} = 1/4$$

$$P_{21} = 3/4 \min\{1, \pi_1/\pi_2\}$$

$$= 3/4 \min\{1, \tfrac{1}{2}\} = 3/8$$

$$P = \begin{bmatrix} \frac{1}{4} & 3/4 \\ 3/8 & 5/8 \end{bmatrix}$$

$$\left[\tfrac{1}{3}, \tfrac{2}{3}\right] \begin{bmatrix} \frac{1}{4} & 3/4 \\ 3/8 & 5/8 \end{bmatrix} = \left[\tfrac{1}{12} + \tfrac{6}{24}, \tfrac{3}{12} + \tfrac{10}{24}\right]$$

$$= \left[\tfrac{1}{3}, 2/3\right]$$