# Markov Chains

Let $\{Y_t\}$ $t = 0, 1, 2, \ldots$

be a sequence of random variables.

$\{Y_t\}$ is a Markov chain if

(a) $Y_t \mid Y_{t-1}, Y_{t-2}, \ldots Y_0$

$= Y_t \mid Y_{t-1}$

(b) $p_t(y_t \mid y_{t-1}) = p(y_t \mid y_{t-1})$

so "p" is the distribution of the next $y$ given the previous $y$.

## Example

$$Y_t \in \{1, 2\}$$

$$P[Y_t = i \mid Y_{t-1} = j] = P_{ji}$$

$$P = [P_{ji}] = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & 3/4 \end{bmatrix}$$

" $P_{ji} = P_{j \to i}$ "

$P(\text{next is } 1 \mid \text{at } 1) = \frac{1}{2}$

$P(\text{next is } 2 \mid \text{at } 2) = 3/4$

## Example AR(1)

$$Y_t = \alpha + \beta Y_{t-1} + \varepsilon_t$$

$$\varepsilon_t \sim N(0, \sigma^2) \text{ iid.}$$

## Stationary Distribution

Suppose $Y_0 \sim \pi$

$p(y_1, y_0) = \pi(y_0)\, p(y_1 | y_0)$

if $Y_1 \sim \pi$ we say

$\pi$ is the stationary

distribution of the markov

chain given by $p(y_t | y_{t-1})$.

(a) draw $Y_0 \sim \pi$

(b) draw $Y_1 \sim p(y_1 | y_0)$

If $Y_1 \sim \pi$, $\pi$ is stationary.

**Example**

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & 3/4 \end{bmatrix}$$

$$\pi = \left[ \tfrac{1}{3},\ 2/3 \right]$$

$$P\{Y_1 = 1\} = P\{Y_0 = 1\} \cdot P\{Y_1 = 1 \mid Y_0 = 1\} + P\{Y_0 = 2\} \cdot P\{Y_1 = 1 \mid Y_0 = 2\}$$

$$= \left(\tfrac{1}{3}\right)\left(\tfrac{1}{2}\right) + (2/3)(1/4)$$

$$= \left(\tfrac{1}{3}\right)\left(\tfrac{1}{2}\right) + \left(\tfrac{1}{3}\right)(2/4) = \frac{1}{3}$$

$$Y_1 \sim \left[\tfrac{1}{3},\ 2/3\right] = \pi$$

**Note** $\quad \pi_1 = \pi P \qquad Y_1 \sim \pi_1$

$$\left[\tfrac{1}{3},\ 2/3\right] \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & 3/4 \end{bmatrix} = \left[\tfrac{1}{3},\ 2/3\right]$$

Note

If $\quad Y_t \sim \pi_t \quad Y_0 \sim \pi$

$$\pi_t = \pi P^t$$

e.g $\quad \pi_2 = \underbrace{(\pi P)}_{\pi_1}(P) = \pi P^2$

## Example

$$y_t = \alpha + \beta y_{t-1} + \varepsilon_t$$

$$\varepsilon_t \sim N(0,1)$$

$$y \sim N\left(\frac{\alpha}{1-\beta}, \frac{\sigma^2}{1-\beta^2}\right) \text{ is stationary.}$$

Suppose $y_0 \sim N\left(\frac{\alpha}{1-\beta}, \frac{\sigma^2}{1-\beta^2}\right)$

$y_1 = \alpha + \beta y_0 + \varepsilon$ is normal.

$$E(y_1) = \alpha + \beta\left[\frac{\alpha}{1-\beta}\right]$$

$$= \frac{\alpha(1-\beta) + \beta\alpha}{1-\beta} = \frac{\alpha}{1-\beta}$$

$$\mathrm{Var}(y_1) = \beta^2 \frac{\sigma^2}{1-\beta^2} + \sigma^2$$

$$= \frac{\sigma^2}{1-\beta^2}$$

## Two Basic Results

Let $Y_0 \sim p_0 \qquad p(y_t \mid y_{t-1})$

$Y_t \sim p_t$

A/ If $\pi$ is stationary,

$$p_t \to \pi \quad \text{as} \quad t \to \infty$$

(no matter what $p_0$ is).

Example

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & 3/4 \end{bmatrix}$$

$$P^t \to \begin{bmatrix} \pi \\ \pi \end{bmatrix} = \begin{bmatrix} \frac{1}{3}, & 2/3 \\ \frac{1}{3}, & 2/3 \end{bmatrix}$$

B

Suppose $\pi$ is the stationary distribution

We want $E(f(Y))$

$Y \sim \pi$

Then $E(f(Y)) \simeq \frac{1}{T} \sum_{t=1}^{T} f(Y_t)$

where $\{Y_t\}$ are generated using the Markov chain.

Example

$$f(y) = \chi_A(y) = \begin{cases} 1 & y \in A \\ 0 & \text{else} \end{cases}$$

$$P(Y \in A) = E[f(Y)]$$

$$\simeq \% \ Y_t \text{ in } A$$

So Histogram of $\{Y_t\}$

is like $\pi$!

# In short

MC:

$$Y_t \sim \pi \quad \text{iid}$$

$$\{Y_t\} \approx \pi$$

MCMC:

$\{Y_t\}$ markov.

$\pi$ is stationary

$$\{Y_t\} \approx \pi$$

Note

If $Y_0 \sim \pi_0$

a very different from

$\pi$ then it may

"take a while" for

the chain to "burn in"

so we may drop some

initial draws.

$$E(f(Y)) \approx \frac{1}{T-T_0} \sum_{t=T_0+1}^{T} f(Y_t)$$