Multinomial Observation and the Dirichlet Conjugate Prior

Observe : $x_i \in \{1, 2, \cdots J\}$

$p = (p_1, p_2, \cdots p_J)$ $\quad 0 \le p_j \le 1$

$\sum p_j = 1$

$$p(x_i = j \mid p) = p_j$$

Convenient Representation of x

Let $y_{ij} = \begin{cases} 1 & x_i = j \\ 0 & \text{else} \end{cases}$ $\quad i = 1, 2, \ldots n$ $\quad j = 1, 2, \ldots J$

$$y_i = (y_{i1}, y_{i2}, \ldots y_{iJ})$$

$x_i \Longleftrightarrow y_i$ one-to-one

$$p(x_i | p) = \prod_j p_j^{y_{ij}}$$

$$= p_j \text{ if } x_i = j$$

Likelihood

$$L(p) \propto \prod_i p(x_i \mid p)$$

$$= \prod_i \prod_j p_j^{y_{ij}}$$

$$= \prod_j p_j^{n_j}$$

$$n_j = (\# \, y_{ij} = 1) = (\# \, x_i = j)$$

Dirichlet Conjugate Prior on p

$$p = (p_1, p_2, \ldots p_J) \qquad 0 \le p_j \le 1$$

$$\sum p_j = 1$$

$$\alpha = (\alpha_1, \alpha_2, \ldots \alpha_J) \qquad \alpha_j > 0$$

$$P(p \mid \alpha) = \frac{\Gamma(\sum \alpha_j)}{\prod \Gamma(\alpha_j)} \prod p_j^{\alpha_j - 1}$$

$$\propto \prod_j p_j^{\alpha_j - 1}$$

Posterior of p

$$p(p \mid x, \alpha) \propto \underset{\text{(likelihood)}}{L(p)} \; \underset{\text{(prior)}}{p(p)}$$

$$\propto \left(\prod_j p_j^{n_j} \right) \left(\prod_j p_j^{\alpha_j - 1} \right)$$

$$\propto \prod_j p_j^{(n_j + \alpha_j) - 1}$$

$$p \mid x, \alpha \sim \text{Dirichlet}((\alpha_j + n_j))$$