Linear Combinations, Covariance and Correlation
Homework Problems
Homework Solutions

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1.1. Problem: Invest-100

Suppose you invest \$100 in an asset.

The return on the asset is the random variable R.

$E(R) = .1.$
$Var(R) = .01.$

Let W be your end of period wealth so that $W = (1 + R)100$.

(a)

What is $E(W)$?

(b)

What is $Var(W)$?

(c)

Suppose R is normal.

Any linear function of a a normal random variable is also normal.

Once you know an RV is normal, all you need is its mean and variance and you know the distribution!

What is the distribution of W?

(d)

What is $\text{Prob}(W > 120)$?

Solution

(a)
$W = 100 + 100\,R, \quad E(W) = 100 + 100\,E(R) = 110.$
(b)
$Var(W) = 100^2\,Var(R) = 100^2\,(.01) = 100.$
(c)
$W \sim N(110, 100).$
(d)
This is Prob($X > \mu + \sigma$) which is .16.

1.2. Problem: Uniform Return

Let U be the uniform random variable which is equally likely to be any value between 0 and 1.

Let R denote a return about which you are uncertain.
Suppose you believe that R is equally likely to be any value between -.1 and .2.

(a) What does the pdf of R look like?

(b) Write R as a linear function of U.

(c)

We know $E(U) = .5$ and $Var(U) = 1/12$.

What is $E(R)$?
What is $Var(R)$?

Solution

(a)
Flat with height $1/.3 = 3.3333$ for $(-.1 < r < .2)$ and 0 everywhere else.
(b)
$R = -.1 + .3\,U$.
(c)
$E(R) = -.1 + .3E(U) = -.1 + .3(.5) = .05.$

$Var(R) = .3^2\ Var(U) = .3^2/12 = .0075.$

2.1. Problem: Covariance and Correlation

Suppose V and W are discrete RVs with joint distribution give by:

		W	
		5	15
V	5	.05	.05
	15	.45	.45

Suppose Y and X are discrete RVs with joint distribution give by:

		X	
		5	15
Y	5	.45	.05
	15	.05	.45

(a) Compute the covariance and correlation between X and Y.

(b) Compute the covariance and correlation between W and V.

(c) Are X and Y independent?

(d) Are W and V independent?

(e)

Are X and Y identically distributed?
Are X and Y iid ?

Solution

(a)
$E(X) = E(Y) = 10.$
$Var(X) = Var(Y) = 25.$

$Cov(X, Y) = .45 * (5 - 10) * (5 - 10) + .05 * (15 - 10) * (5 - 10) + .05 * (5 - 10) * (15 - 10) + .45 * (15 - 10) * (15 - 10) = 20.$

$Cor(X, Y) = Cov(X, Y)/(\sigma_Y \sigma_X) = 20/25 = .8.$
(b)
$E(W) = 10,\ Var(W) = 25.$

$E(V) = .1 * 5 + .9 * 15 = 14.$
$Var(V) = .1 * (5 - 14)^2 + .9 * (15 - 14)^2 = 9.$

$Cov(W, V) = .05 * (5 - 10) * (5 - 14) + .05 * (15 - 10) * (5 - 14) + .45 * (5 - 10) * (15 - 14) + .45 * (15 - 10) * (15 - 14) = 0.$

Since the covariance is 0, the correlation is 0.

(c)
The covariance is not 0, so the cannot be independent.
Clearly $P(Y = 15|X = 15) = .9 > P(Y = 15|X = 5) = .1!$
(d)
Given $V = 5$, W is 5 or 15, each with prob .5.
Given $V = 15$, W is 5 or 15, each with prob .5.
Thus, they are independent.
The 0 covariance is not enough to be sure they are independent!
(e)
X and Y have the same marginals (identically distributed) but they are not independent so they are not iid.

3.1. Problem: Portfolio Return, Three Risky Assets

Let R_f denote the return on a riskless asset (you get return R_f for sure!).
Suppose $R_f = .02$.

Suppose R_1, R_2, and R_3 are random variables representing returns on three risky assets.

$E(R_1) = .05$, $E(R_2) = .1$, $E(R_3) = .15$.

For each risky asset, the standard deviation equals the mean.

The correlation between R_1 and R_2 is .5 and R_3 is independent of R_1 and R_2.

Suppose $P_1 = .4\,R_f + .6\,R_3$.

(a) What is $E(P_1)$?

(b) What is $Var(P_1)$?

(c) What is the correlation between P_1 and R_3?

Suppose $P_2 = .2\,R_f + .4\,R_1 + .4\,R_2$.

(d) What is $E(P_2)$?

(e) What is $Var(P_2)$?

(f) What is the correlation between P_1 and P_2?

Solution

(a)
.4*.02 + .6*.15 = .098.
(b)
$(.6^2) * (.15^2) = .0081$.
(c)
1 since they are perfectly linearly related.
(d)
.2*.02 + .4*.05 + .4*.1 = .064.

(e)
First we need $Cov(R_1, R_2)$.
We are given the correlation and the standard deviations.

So,
cov = cor(product of sds) = .5*.05*.1 = .0025.

$Var(P_2) =$
$(.4^2) * (.05^2) + (.4^2) * (.1^2) + 2 * .4 * .4 * .0025 = .0028.$
(f)
Since R_3 is independent of R_1 and R_2, P_1 is independent of P_2.
Hence, the correlation is 0.

3.2. Problem: Sum and Average of IID

Suppose we can play a game with gives a random payout.
Let W_i denote the payout on the i^{th} play of the game.
The W_i are iid, with $E(W_i) = 0$, $Var(W_i) = 10$.
(a)

Suppose there is no cover charge and you play the game the first two times so that your total winnings are

$$T = W_1 + W_2.$$

What are $E(T)$ and $Var(T)$?

(b)

Suppose you are interested in your average winnings:

$$\bar{W} = \frac{W_1 + W_2}{2} = \frac{1}{2} W_1 + \frac{1}{2} W_2.$$

What are $E(\bar{W})$ and $Var(\bar{W})$?

(c)

Now suppose you play 100 times.

$$T = W_1 + W_2 + \ldots + W_{100} = \sum_{i=1}^{100} W_i.$$

$$\bar{W} = \frac{1}{100}(W_1 + W_2 + \ldots + W_{100}) = \frac{1}{100} \sum_{i=1}^{100} W_i.$$

What are $E(T)$ and $Var(T)$?
What are $E(\bar{W})$ and $Var(\bar{W})$?

Solution

(a)
$E(T) = E(W_1) + E(W_2) = 2 * 0 = 0.$
$Var(T) = Var(W_1) + Var(W_2) = 2 * 10 = 20.$
(b)
$E(\bar{W}) = \frac{1}{2}E(W_1) + \frac{1}{2}E(W_2) = .5 * 0 + .5 * 0 = 0.$
$Var(\bar{W}) = \frac{1}{2^2}Var(W_1) + \frac{1}{2^2}Var(W_2) = \frac{1}{2^2}(2 * 10) = \frac{10}{2} = 5.$
Alternatively, you could notice that $\bar{W} = \frac{1}{2}T$, so that
$Var(\bar{W}) = \frac{1}{2^2}Var(T) = 20/4 = 5.$
(c)
$E(T) = 0,\ Var(T) = 100 * 10 = 1000.$
$E(\bar{W}) = (\frac{1}{100})(100 * 0) = 0.$
$Var(\bar{W}) = \frac{10}{100} = .1$

(d)

$$E(\bar{X}) = \mu, \quad Var(\bar{X}) = \frac{\sigma^2}{n}.$$